

# FROM FREUDENTHAL'S SPECTRAL THEOREM TO PROJECTABLE HULLS OF UNITAL ARCHIMEDEAN LATTICE-GROUPS, THROUGH COMPACTIFICATIONS OF MINIMAL SPECTRA

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**ABSTRACT.** We use a landmark result in the theory of Riesz spaces — Freudenthal's 1936 Spectral Theorem — to canonically represent any Archimedean lattice-ordered group  $G$  with a strong unit as a (non-separating) lattice-group of real valued continuous functions on an appropriate  $G$ -indexed zero-dimensional compactification  $w_G Z_G$  of its space  $Z_G$  of *minimal* prime ideals. The two further ingredients needed to establish this representation are the Yosida representation of  $G$  on its space  $X_G$  of *maximal* ideals, and the well-known continuous surjection of  $Z_G$  onto  $X_G$ . We then establish our main result by showing that the inclusion-minimal extension of this representation of  $G$  that separates the points of  $Z_G$  — namely, the sublattice subgroup of  $C(Z_G)$  generated by the image of  $G$  along with all characteristic functions of clopen (closed and open) subsets of  $Z_G$  which are determined by elements of  $G$  — is precisely the classical projectable hull of  $G$ . Our main result thus reveals a fundamental relationship between projectable hulls and minimal spectra, and provides the most direct and explicit construction of projectable hulls to date. Our techniques do require the presence of a strong unit.

## 1. INTRODUCTION

In 1936, Freudenthal proved his well-known Spectral Theorem [11] for Riesz spaces (real linear spaces with a compatible lattice order) with motivations coming from the theory of integration. (See [17, 40.2] for a handbook treatment.)

In its basic version, the theorem asserts that any element of a Riesz space  $R$  with a strong unit  $u$  and the principal projection property may be uniformly approximated, in the norm that  $u$  induces on  $R$ , by abstract characteristic functions — “components of the unit  $u$ ”. See Subsection 2.1 for more details. Freudenthal's theorem led to a considerable amount of research on Riesz spaces and their generalisations, the lattice-ordered Abelian groups that concern us here, and which we call  $\ell$ -groups for short. (For background we refer to [17, 10, 13].) One main line of research concentrated on extending one given structure  $G$  to a minimal completion that enjoys the principal projection property, where Freudenthal's theorem therefore applies. Such an extension is called the *projectable hull* of  $G$ ; please see Subsection 2.2 for details.

In 1973, Conrad [7] proved the existence and uniqueness of projectable hulls of (a class of lattice-groups more general than) Archimedean  $\ell$ -groups, using his previous

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construction in [6] of the *essential closure* of such an  $\ell$ -group — the largest extension of the structure that is *essential*, in the sense recalled in Subsection 2.2. At about the same time, Chambless [5] exhibited a different construction of the projectable hull based on direct limits; cf. also Bleier’s construction in [4]. Here we present a new construction of the projectable hull of an Archimedean  $\ell$ -group equipped with a strong order unit  $u$  — an element whose multiples eventually dominate any other element in the  $\ell$ -group — that does not use direct limits, nor essential closures. Our construction exposes instead the intimate connection between projectable hulls and zero-dimensional compactifications of spectral spaces of minimal prime ideals. Closing the circle of ideas beginning with Freudenthal, to establish this connection we will need to apply his Spectral Theorem at a key step of the construction. We now recall some standard notions, and introduce notations that will remain in force throughout the paper.

Throughout, all lattice-ordered groups are Abelian, and referred to simply as  $\ell$ -groups for short. We write  $\mathbf{U}$  for the category whose typical object is a pair  $(G, u)$ , where  $G$  is an  $\ell$ -group that is *Archimedean* — whenever  $0 \leq ng \leq h$  for  $h, g \in G$  and all integers  $n \geq 1$ , then  $g = 0$  — equipped with a distinguished (*strong order*) unit  $u \in G$  — an element  $u \geq 0$  such that for all  $g \in G$  there is an integer  $n \geq 1$  such that  $nu \geq g$ . As morphisms, we take the lattice-group homomorphisms ( $\ell$ -homomorphisms) that are *unital*, i.e. preserve the distinguished units. It will transpire that our techniques do require the existence of a strong unit, as opposed to the existence of a weak unit. Recall that a *weak (order) unit* of  $G$  is an element  $w \in G^+$  such that for each  $g \in G$ ,  $w \wedge |g| = 0$  implies  $g = 0$ . Here,  $|g| := (g \vee 0) + (-g \vee 0)$  is the *absolute value* of  $g$ .

By an *ideal* in an  $\ell$ -group we mean, as usual, a sublattice subgroup  $I$  of  $G$  that is *order-convex*: whenever  $a, c \in I$ ,  $b \in G$ , and  $a \leq b \leq c$ , then  $b \in I$ . Ideals are exactly the kernels of (unital)  $\ell$ -homomorphisms, i.e. the morphisms in the unrestricted category of abelian  $\ell$ -groups, and the usual homomorphism theorems hold. An ideal  $\mathfrak{p}$  of  $G$  is *prime* if, and only if, it is proper ( $\mathfrak{p} \neq G$ ) and the quotient  $\ell$ -group  $G/\mathfrak{p}$  is totally ordered. A prime ideal is *maximal* if it is inclusion-maximal — equivalently, if  $G/\mathfrak{p}$  is non-trivial and *simple*, i.e. it has no non-trivial proper ideals. Ideals that are inclusion-maximal are automatically prime. A prime ideal is *minimal* if it is inclusion-minimal. For any unital  $\ell$ -group  $(G, u)$ , we denote by  $\text{Max } G$  the collection of its maximal (prime) ideals, and by  $\text{Min } G$  the collection of its minimal prime ideals. We topologize both  $\text{Max } G$  and  $\text{Min } G$  using the *spectral*, or *Zariski* topology. The closed sets for this topology are given by subsets of the form

$$\mathbb{V}_M(A) := \{\mathfrak{m} \in \text{Max } G \mid \mathfrak{m} \supseteq A\}$$

and

$$\mathbb{V}_m(A) := \{\mathfrak{p} \in \text{Min } G \mid \mathfrak{p} \supseteq A\},$$

as  $A$  ranges over arbitrary subsets of  $G$ . The resulting topological spaces are called the *maximal* and *minimal prime spectrum* of  $G$ , respectively. The topology on  $\text{Max } G$  is also called the *hull-kernel* topology, because it agrees with the classical hull-kernel topology for rings of continuous functions [12], *mutatis mutandis*. Accordingly, we call  $\mathbb{V}_M(A)$  (or  $\mathbb{V}_m(A)$ ) the *zero set* of  $A$  (on the appropriate space), and its complement — denoted by  $\mathbb{S}_M(A)$  (or  $\mathbb{S}_m(A)$ ) — the *support* of  $A$ . It is

known [3, 10.2.1] that the collections

$$\{\mathbb{V}_M(\{g\})\}_{g \in G} \quad \text{and} \quad \{\mathbb{V}_m(\{g\})\}_{g \in G}$$

form a closed base for  $\text{Max } G$  and  $\text{Min } G$ , respectively. Throughout we write  $\mathbb{V}_M(g)$  in place of  $\mathbb{V}_M(\{g\})$ , and similarly for  $\mathbb{V}_m$ ,  $\mathbb{S}_M$ , and  $\mathbb{S}_m$ .

The space  $\text{Max } G$  is a Hausdorff space that is compact precisely because of the assumption that  $G$  has a (strong) unit  $u$ ; see [3, 10.2.5]. The space  $\text{Min } G$  is a Hausdorff zero-dimensional space that need not be compact [3, 10.2.1]. Whether it is or not has nothing to do with the existence of a strong unit, but rather with complementation properties of the lattice  $G^+ := \{g \in G \mid g \geq 0\}$ , the *positive cone* of  $G$ ; see Section 3.

**Notation.** For the rest of this paper, we let  $(G, u)$  denote a  $\mathbf{U}$ -object, and set

$$\begin{aligned} X_G &:= \text{Max } G, \\ Z_G &:= \text{Min } G. \end{aligned}$$

If  $X$  is any topological space, always at least Tychonoff, we write  $C(X)$  for the  $\ell$ -group of continuous functions  $X \rightarrow \mathbb{R}$  under pointwise operations. If  $X$  is compact, the function  $1_X$  constantly equal to 1 over  $X$  is a strong unit of  $C(X)$  by the Extreme Value Theorem. We always tacitly consider  $C(X)$  endowed with the distinguished unit  $1_X$ , and hence as a  $\mathbf{U}$ -object when  $X$  is compact Hausdorff. The classical *Yosida representation* [21] of  $(G, u)$  yields a canonical unital lattice-group embedding  $\hat{\cdot}: G \hookrightarrow C(X_G)$ ; details are recalled in Subsection 2.3.

It is well known that  $Z_G$  is canonically thrown onto  $X_G$ , as follows. Given  $\mathfrak{a} \in Z_G$ , a standard argument [17, 27.4] shows that, by virtue of the presence of the (strong) unit  $u$ , there exists at least one  $\mathfrak{m}_{\mathfrak{a}} \in X_G$  such that  $\mathfrak{a} \subseteq \mathfrak{m}_{\mathfrak{a}}$ . Since the prime ideals of  $G$  form a *root system* under set-theoretic inclusion [3, 2.4.3] — that is, the set of prime ideals containing any given prime ideal is linearly ordered — such an  $\mathfrak{m}_{\mathfrak{a}}$  must be unique; in other words, the set  $\uparrow \mathfrak{a} \cap X_G$  is a singleton, where  $\uparrow \mathfrak{a} := \{\mathfrak{b} \subseteq G \mid \mathfrak{a} \subseteq \mathfrak{b}, \mathfrak{b} \text{ a prime ideal}\}$ . Hence there is a function

$$\lambda: Z_G \twoheadrightarrow X_G \tag{1}$$

defined by

$$\mathfrak{a} \in Z_G \xrightarrow{\lambda} \mathfrak{m}_{\mathfrak{a}} \in X_G. \tag{2}$$

By [3, 10.2.5], the map  $\lambda$  is continuous, and is a surjection by the standard fact that each prime ideal contains a minimal prime ideal [3, 2.4.5].

**Remark 1.1.** The strong unit is crucial here. There exist non-trivial Archimedean (and even Dedekind-complete)  $\ell$ -groups with a weak unit and no maximal ideal at all; see [17, 27.8]. Thus, while the Yosida representation has an important extension to  $\ell$ -groups with a weak unit [22, 15], the existence of the map (1), a key ingredient to our construction, does require a strong unit. A generalisation of our results to the case of weak units would surely be of interest, but it would require substantial modifications.

Composition of the map  $\lambda$  with the Yosida representation of  $G$  embeds  $G$  as a unital sublattice-subgroup ( $\ell$ -subgroup) into  $C(Z_G)$ : one sends  $g \in G$  to  $\hat{g} \circ \lambda: Z_G \rightarrow \mathbb{R}$ . The assignment is injective because  $\lambda$  is surjective. In Section 3 this observation is considerably strengthened. It turns out that  $G$  determines a specific zero-dimensional compactification of its minimal spectrum which we denote

$w_G Z_G$ . The clopen subsets of  $w_G Z_G$ , we will see, are (finitely) generated from the clopen subsets of  $Z_G$  of the form  $\mathbb{V}_m(g)$ . (Please see Section 3 for details on this compactification.) We prove in Theorem 3.11 that  $G$  embeds as a unital  $\ell$ -subgroup of  $C(w_G Z_G)$ . We will see that this stronger embedding of  $G$  is in fact granted by Freudenthal's Spectral Theorem. Now, by the Yosida theory (see again Subsection 2.3), the image of  $G$  in  $C(w_G Z_G)$  does not separate the points of the base space, unless  $X_G$  and  $w_G Z_G$  are homeomorphic. We can however consider a minimal extension of the image of  $G$  inside  $C(w_G Z_G)$  which separates the points. Indeed, since  $w_G Z_G$  is zero-dimensional, there is a canonical such extension: we must adjoin to the image of  $G$  all characteristic functions of clopen subsets of  $w_G Z_G$ . We thereby obtain a unital embedding

$$\pi_G: G \hookrightarrow \mathcal{P}(G), \quad (3)$$

where  $\mathcal{P}(G)$  denotes the unital  $\ell$ -subgroup of  $C(w_G Z_G)$  generated by the representation of  $G$  into  $C(w_G Z_G)$ , together with all *characteristic functions*  $w_G Z_G \rightarrow \mathbb{R}$  — the continuous maps with range contained in  $\{0, 1\}$ . We now have the homeomorphism  $\text{Max } \mathcal{P}(G) \cong w_G Z_G$ . In Theorem 4.5 we show that the elements of  $\mathcal{P}(G)$  may be characterised amongst elements of  $C(w_G Z_G)$  as those functions with the property that, for an appropriate finite partition of  $w_G Z_G$  into clopens, they agree with the image of some element of  $G$  locally at each clopen. Building on this we finally show in Theorem 5.1 that (3) is the projectable hull of  $(G, u)$ , thus obtaining our main result. Summarising, we prove the existence of the projectable hull of any **U**-object  $(G, u)$  by exhibiting it as a natural substructure of  $C(w_G Z_G)$ , namely,  $\mathcal{P}(G)$ .

Several intermediate results in this paper admit a fuller development of considerable potential interest. We focus here on the proof of our main Theorem 5.1, and postpone further results to future work.

## 2. PRELIMINARIES

**2.1. Polars and projection properties.** For the standard notions that we recall in this subsection, see [3, Ch. 3, 6, and 11], together with [17, Ch. 4, §24]. Given any  $\ell$ -group  $A$ , the elements  $x, y \in A$  are *orthogonal*, written  $x \perp y$ , if  $|x| \wedge |y| = 0$ . For  $T \subseteq A$ , we set

$$T^\perp := \{x \in A \mid x \perp y \text{ for all } y \in T\};$$

we write  $T^{\perp\perp}$  instead of  $(T^\perp)^\perp$ , and  $x^\perp$  instead of  $\{x\}^\perp$  for  $x \in A$ . A subset  $S \subseteq A$  is a *polar* if it satisfies  $S = S^{\perp\perp}$ , or equivalently, if there exists  $T \subseteq A$  such that  $S = T^\perp$ . When necessary, we will denote polars computed in  $A$  by  $T^{\perp_A}$ . We write  $\text{Pol } A$  to denote the set of polars of  $A$ . Under the inclusion order,  $\text{Pol } A$  is a complete distributive lattice with  $A = 0^\perp$  as maximum,  $\{0\} = A^\perp = 0^{\perp\perp}$  as minimum, meets given by intersections, and joins given by  $\bigvee S_i := (\bigcup S_i)^{\perp\perp}$ . It can be shown that  $\text{Pol } A$  is a complete Boolean algebra, with complementation given by the map  $S \in \text{Pol } A \mapsto S^\perp \in \text{Pol } A$ . In particular, for any subset  $T \subseteq A$  we have  $T^{\perp\perp\perp} = T^\perp$ .

If  $x \in A$ , the set  $x^{\perp\perp}$  is called the *principal polar* generated by  $x$ . Then  $x^{\perp\perp} \in \text{Pol } A$ , and  $x^{\perp\perp} = \bigcap_{x \in S \in \text{Pol } A} S$ , that is,  $x^{\perp\perp}$  is the inclusion-smallest polar containing  $\{x\}$ . We write  $\text{Pol}_p A$  to denote the set of principal polars of  $A$ ; it

is a sublattice of  $\text{Pol } A$ , because of the identities

$$(x \wedge y)^{\perp\perp} = x^{\perp\perp} \cap y^{\perp\perp} \quad (4)$$

$$(x \vee y)^{\perp\perp} = x^{\perp\perp} \vee y^{\perp\perp}, \quad (5)$$

which hold for each  $x, y \in A^+$ . Further, the minimum  $0^{\perp\perp}$  of  $\text{Pol } A$  lies in  $\text{Pol}_p A$ . However, the maximum  $A = 0^{\perp}$  of  $\text{Pol } A$  need not be a principal polar: in fact, this happens precisely when  $A$  has a weak unit  $w$ , and in that case  $A = w^{\perp\perp}$ . Even when  $A$  has a weak unit,  $\text{Pol}_p A$  may fail to be a Boolean subalgebra of  $\text{Pol } A$ , because the complement of a principal polar need not be principal.

An ideal  $I \subseteq A$  is *closed*, or is a *band*, if for each  $S \subseteq I$  such that  $\bigvee S$  exists in  $A$ , we have  $\bigvee S \in I$ . It can be shown that each polar is a band; for the converse, we have the important

**Lemma 2.1.** *An  $\ell$ -group  $A$  is such that its polars coincide with its bands if, and only if,  $A$  is Archimedean.*

*Proof.* [3, 11.1.10]. □

A band  $I \subseteq A$  is a *projection band* if there is a product splitting  $A \cong I \times I^{\perp}$ .

**Definition 2.2** (Cf. [17, 24.8]). An  $\ell$ -group  $A$  is said to have the *principal projection property*, or to be *projectable*, if each principal band of  $A$  is a projection band. Further,  $A$  is said to have the *projection property*, or to be *strongly projectable*, if each band of  $G$  is a projection band.

We recall here a standard fact:

**Lemma 2.3.** *An  $\ell$ -group with the principal projection property must be Archimedean.*

*Proof.* The (easy) proof for vector lattices given in [17, 24.9] works for  $\ell$ -groups without changes. □

**Remark 2.4.** Projection properties are a classical topic in the theory of vector lattices, see [17, Ch. 4]. In the literature on  $\ell$ -groups, it is standard to call  $A$  *projectable* when each of its principal polars is a cardinal summand (i.e. a factor of a product splitting) of  $A$ , and *strongly projectable* when the same holds for all polars. Thus, we see from Lemmas 2.3 and 2.1 that an  $\ell$ -group  $A$  has the principal projection property if, and only if, it is projectable in the present sense; and that it has the projection property if, and only if, it is strongly projectable in the present sense. Cf. also [3, 7.5]. This explains the alternative terminologies in Definition 2.2. In the rest of this paper we shall use the terminology *projectable*.

A *component of the unit*  $u$  is an element  $\chi \in G$  such that  $\chi \vee (u - \chi) = u$  and  $\chi \wedge (u - \chi) = 0$ . It is well known that this entails the existence of a product splitting  $G \cong \chi^{\perp} \times \chi^{\perp\perp}$ . Conversely, if  $G \cong A \times B$  in  $\mathbf{U}$ , then there is a unique  $\chi \in G$  — namely, the image in  $G$  of the unit of  $B$  under the unital isomorphism  $G \cong A \times B$  — that is a component of the unit  $u$  such that  $A \cong \chi^{\perp}$  and  $B \cong \chi^{\perp\perp}$ . We use these elementary facts without further justification throughout.

Finally, we recall the version of Freudenthal's Spectral Theorem that we will use.

**Theorem 2.5.** *Let  $R$  be a Riesz space that is projectable and has a unit  $u$ . For  $v \in R$ , set  $\|v\|_u := \inf \{\lambda \in \mathbb{R} \mid \lambda \geq 0 \text{ and } \lambda u \geq |v|\}$ . Then  $\|v\|_u$  is a norm on  $R$ . For each  $v \in R$  there is a sequence  $\{c_i\}_{i \geq 1} \subseteq R$  of linear combinations of components of  $u$  that converges to  $v$  uniformly in the norm  $\|\cdot\|_u$ .*

*Proof.* See [17, 40.2].  $\square$

**2.2. Essential extensions and the projectable hull.** A monomorphism  $\iota: (G, u) \hookrightarrow (H, v)$  in  $\mathbf{U}$  will be referred to as an *extension* (of  $G$  by  $H$ ). The extension is *essential* if whenever a  $\mathbf{U}$ -morphism  $f: (H, v) \rightarrow (A, a)$  is such that the composition  $f \circ \iota$  is monic, then  $f$  is monic. Amongst several well-known characterisations of essential extensions we shall use the following.

**Lemma 2.6.** *Let  $\iota: (G, u) \hookrightarrow (H, v)$  be a monomorphism in  $\mathbf{U}$ . The following are equivalent.*

- (1) *The extension  $\iota$  is essential.*
- (2) *The map  $\nu_H: P \in \text{Pol } H \mapsto P \cap \iota(G) \in \text{Pol } \iota(G)$  is an isomorphism from the Boolean algebra of polars of  $H$  onto that of  $\iota(G)$ . The inverse isomorphism is the map  $\nu_H^{-1}: Q \in \text{Pol } \iota(G) \mapsto Q^{\perp \perp H} \in \text{Pol } H$ .*
- (3) *For each  $y \in H$  with  $y > 0$  there is  $x \in G$  with  $0 < \iota(x) < ny$  for some integer  $n > 0$ .*

*Proof.* See [6, Prop. 3.1 and Thm. 3.7] and [7, §2].  $\square$

**Definition 2.7.** An essential extension  $\epsilon: (G, u) \hookrightarrow (K, w)$  in  $\mathbf{U}$  is said to be a *projectable hull* if  $K$  is projectable, and whenever  $\iota: (G, u) \hookrightarrow (H, v)$  is another essential extension with  $H$  projectable, there exists an injective  $\ell$ -homomorphism  $\varphi: (K, w) \rightarrow (H, v)$  in  $\mathbf{U}$  that makes the following diagram commute.

$$\begin{array}{ccc} (G, u) & \xhookrightarrow{\epsilon} & (K, w) \\ & \searrow \iota & \downarrow \varphi \\ & & (H, v) \end{array}$$

It turns out that the  $\ell$ -homomorphism  $\varphi$  in the preceding definition is automatically an essential extension. Also note that a projectable hull is unique up to an isomorphism in  $\mathbf{U}$ .

**Remark 2.8.** Through the general treatment in [2], hulls related to projectability properties can and have been fruitfully investigated at the level of all lattice-ordered (not necessarily Abelian) groups, with no assumption on the existence of units. In particular, any lattice-ordered group turns out to have a strongly projectable hull in this generalised sense, [2, Thm. 2.25], which agrees with the usual one in the representable case.

**2.3. The Yosida representation: the case of strong units.** For  $X$  a topological space, recall that a subset  $S \subseteq C(X)$  is said to *separate the points of  $X$*  if for any  $x \neq y \in X$  there is  $f \in S$  with  $f(x) \neq f(y)$ . The next result summarises the classical Yosida representation; everything is rooted and essentially proved in [21].

**Theorem 2.9** (The Yosida Representation). *Recall that  $(G, u)$  is a  $\mathbf{U}$ -object with maximal spectral space  $X_G$ .*

- (a) *For each  $\mathfrak{m} \in X_G$ , there exists a unique monomorphism*

$$\iota_{\mathfrak{m}}: (G/\mathfrak{m}, u/\mathfrak{m}) \hookrightarrow (\mathbb{R}, 1)$$

*in  $\mathbf{U}$ . Upon setting*

$$\widehat{g}(\mathfrak{m}) := \iota_{\mathfrak{m}}(g/\mathfrak{m}) \in \mathbb{R},$$

each  $g \in G$  induces a function

$$\widehat{g}: X_G \rightarrow \mathbb{R}$$

that is continuous with respect to the spectral topology on the domain and the Euclidean topology on the co-domain.

(b) The map

$$\widehat{\cdot}: (G, u) \longrightarrow (C(X_G), 1_{X_G})$$

given by (a) is a monomorphism in  $\mathbf{U}$  whose image  $\widehat{G} \subseteq C(X_G)$  separates the points of  $X_G$ .

(c)  $X_G$  is unique up to a unique homeomorphism with respect to its properties. More explicitly, if  $Y$  is any compact Hausdorff space, and  $e: (G, u) \hookrightarrow (C(Y), 1_Y)$  is any monomorphism in  $\mathbf{U}$  whose image  $e(G) \subseteq C(Y)$  separates the points of  $Y$ , then there exists a unique homeomorphism  $f: Y \rightarrow X_G$  such that  $(e(g))(y) = \widehat{g}(f(y))$  for all  $g \in G$  and  $y \in Y$ .

**Remark 2.10.** For the more general Yosida representation in the category of  $\ell$ -groups equipped with a weak unit, see the standard reference [15].

**Remark 2.11.** Let us explicitly observe that components of the unit  $1_{X_G}$  in  $C(X_G)$  are precisely the characteristic functions  $X_G \rightarrow \mathbb{R}$ , i.e. the continuous functions with range contained in  $\{0, 1\}$ .

### 3. REPRESENTING AN $\ell$ -GROUP ON ITS MINIMAL SPECTRUM

Recall that  $Z_G$  denotes the minimal spectral space of the  $\mathbf{U}$ -object  $(G, u)$ . We show in this section that  $G$  may be represented as an  $\ell$ -subgroup of  $C(w_G Z_G)$  for the zero-dimensional Wallman compactification  $w_G Z_G$  of  $Z_G$ , which we describe below. Before dealing with the general case, let us pause to recall that compactness of  $Z_G$  is equivalent to a complementation property of  $G$ .

**Definition 3.1** ([9]). An  $\ell$ -group  $A$  is *complemented* if for each  $x \in A$  there exists  $y \in A$  such that  $|x| \wedge |y| = 0$  and  $|x| \vee |y| \neq 0$  is a weak unit of  $A$ .

Throughout we write  $\cdot \setminus \cdot$  for set-theoretic difference. We recall  $S_m = \text{Min } A \setminus \mathbb{V}_m(g)$ . Conrad and Martinez [8] proved the equivalence of (i) and (ii) below.

**Lemma 3.2.** *For an  $\ell$ -group  $A$ , the following are equivalent.*

- (i)  $A$  is complemented.
- (ii)  $\text{Min } A$  is compact.
- (iii) There exists a weak unit in  $A$ , the lattice  $\text{Pol}_{\mathbf{p}} A$  is bounded, and the inclusion map  $\text{Pol}_{\mathbf{p}} A \hookrightarrow \text{Pol } A$  is a homomorphism of Boolean algebras.
- (iv) The distributive lattice  $(\{\mathbb{V}_m(g)\}_{g \in A}, \cap, \cup)$  is a Boolean algebra, i.e. for all  $g \in A$  there is  $f \in A$  such that  $S_m(g) = \mathbb{V}_m(f)$ .

*Proof.* (i  $\Leftrightarrow$  ii) is [8, 2.2]. (i  $\Leftrightarrow$  iii) is also stated in passing in [8]; its proof is an elementary application of (4-5).

To prove (iii  $\Leftrightarrow$  iv) we recall that  $\bigcap_{\mathbf{p} \in \text{Min } G} \mathbf{p} = \{0\}$  and that a prime  $\mathbf{p} \in \text{Spec } A$  is minimal if and only if for all  $g \in \mathbf{p}$  there is an element  $h \notin \mathbf{p}$  with  $h \perp g$  (see [3, 3.4.13]). We first show  $\mathbb{V}_m(g) = \mathbb{V}_m(g^{\perp\perp})$  for every  $g \in A$ . Since  $g \in g^{\perp\perp}$ ,  $\mathbb{V}_m(g^{\perp\perp}) \subseteq \mathbb{V}_m(g)$ . Let  $\mathbf{p} \in \mathbb{V}_m(g)$ . Since  $g \in \mathbf{p}$  and  $\mathbf{p} \in \text{Min } A$ , there exists  $h \notin \mathbf{p}$  such that  $g \perp h$ , whence  $h \in g^{\perp}$ . Let  $f \in g^{\perp\perp}$ . Then  $f \in \mathbf{p}$ , because  $|f| \wedge |h| = 0$  and  $\mathbf{p}$  is prime. This ensures  $\mathbb{V}_m(g^{\perp\perp}) \subseteq \mathbb{V}_m(g)$ .

(iv  $\Rightarrow$  iii) Let  $g \in A$  and  $f \in A$  such that  $\mathbb{S}_m(g) = \mathbb{V}_m(f)$ . Then each  $\mathfrak{p} \in \text{Min } A$  is contained either in  $\mathbb{V}_m(g)$  or in  $\mathbb{V}_m(f)$ . We show that  $g^{\perp\perp} = f^{\perp}$ .

Let  $a \in f^{\perp}$ . By primality,  $a \in \mathfrak{p}$  for each  $\mathfrak{p} \in \mathbb{V}_m(g) = \mathbb{S}_m(f)$ , and hence  $\mathbb{V}_m(g) \subseteq \mathbb{V}_m(a)$ . For each  $b \in g^{\perp}$ ,  $b \in \mathfrak{p}$  whenever  $g \notin \mathfrak{p}$ , whence  $b \in \mathfrak{p}$  for each  $\mathfrak{p} \in \mathbb{V}_m(f)$ . Therefore,  $|a| \wedge |b| \in \mathfrak{p}$  for each  $\mathfrak{p} \in \text{Min } A$ , and hence  $|a| \wedge |b| = 0$ . This ensures  $a \in g^{\perp\perp}$ , whence  $g^{\perp\perp} \supseteq f^{\perp}$ .

For the other inclusion, let  $a \in g^{\perp\perp}$ . By way of contradiction, suppose  $|a| \wedge |f| = c > 0$ . Then there exists  $\mathfrak{p}_c \in \text{Min } A$  such that  $c \notin \mathfrak{p}_c$ . As a consequence,  $a \notin \mathfrak{p}_c$ ,  $f \notin \mathfrak{p}_c$ , and  $g \in \mathfrak{p}_c$ . By minimality of  $\mathfrak{p}$ , there exists  $b \notin \mathfrak{p}_c$  such that  $b \perp g$ . Since  $|a| \wedge |b| = 0 \in \mathfrak{p}_c$ , then  $a \in \mathfrak{p}_c$ , a contradiction. Hence we proved  $g^{\perp\perp} \subseteq f^{\perp}$ .

As a consequence,  $g^{\perp\perp\perp} = f^{\perp\perp}$ , whence the principal polar  $f^{\perp\perp}$  is the complement of  $g^{\perp\perp}$  in  $\text{Pol}_p A$ . Moreover, by taking  $g = 0$ , there exists  $w \in A$  such that  $A = 0^{\perp\perp\perp} = w^{\perp\perp}$ . Then  $\text{Pol}_p A$  is bounded and a Boolean algebra, and, as a direct consequence of the definition,  $w$  is a weak unit for  $A$ .

(iii  $\Rightarrow$  iv) Let  $g \in A$  and  $f \in A$  such that  $f^{\perp\perp}$  is the complement of  $g^{\perp\perp}$  in  $\text{Pol}_p A$ . Then  $g^{\perp\perp\perp} = f^{\perp\perp}$ , and  $g^{\perp\perp} = f^{\perp}$ . We show  $\mathbb{V}_m(f) = \mathbb{S}_m(f^{\perp})$ , whence  $\mathbb{V}_m(f) = \mathbb{S}_m(g^{\perp\perp}) = \mathbb{S}_m(g)$ . Let  $\mathfrak{p} \in \mathbb{V}_m(f)$ , then  $f \in \mathfrak{p}$ . Since  $\mathfrak{p}$  is minimal, there is  $h \in A$  such that  $h \perp f$  and  $h \notin \mathfrak{p}$ . Hence  $h \in f^{\perp}$  and  $\mathfrak{p} \notin \mathbb{V}_m(f^{\perp})$ . This ensures  $\mathbb{V}_m(f) \subseteq \mathbb{S}_m(f^{\perp})$ . For the other inclusion, let  $\mathfrak{p} \in \mathbb{S}_m(f^{\perp})$  and  $h \notin \mathfrak{p}$  such that  $h \perp f$ . Since  $|h| \wedge |f| = 0$  the primality of  $\mathfrak{p}$  ensures  $f \in \mathfrak{p}$ , whence  $\mathfrak{p} \in \mathbb{V}_m(f)$ . Therefore  $\mathbb{S}_m(f^{\perp}) \subseteq \mathbb{V}_m(f)$  and the proof is complete.  $\square$

**Remark 3.3.** Versions of Lemma 3.2 for commutative rings, distributive lattices, and vector lattices were proved in [16, 3.4], [19, Prop. 3.2], and [17, 37.4], respectively.

We now turn to the  $w_G$ -compactification. Recall [18, 4.4(a)] that a *Wallman base* of a Hausdorff space  $X$  is a base  $\mathcal{L}$  of closed sets for  $X$  that is stable under finite intersections and unions (and thus contains, in particular,  $\emptyset$  and  $X$ ), is such that if  $A \in \mathcal{L}$  and  $x \in X \setminus A$  then there is  $B \in \mathcal{L}$  with  $x \in B$  and  $A \cap B = \emptyset$ , and is such that for  $A, B \in \mathcal{L}$  satisfying  $A \subseteq X \setminus B$  there exist  $C, D \in \mathcal{L}$  with  $A \subseteq X \setminus C \subseteq D \subseteq X \setminus B$ . Given such a base  $\mathcal{L}$ , let  $w_{\mathcal{L}}X$  denote the collection of inclusion-maximal lattice filters of  $\mathcal{L}$ . The collection of sets  $\{\mathcal{F} \in w_{\mathcal{L}}X \mid A \in \mathcal{F}\}$ , as  $A$  ranges in  $\mathcal{L}$ , is a base for the closed sets of a topology on  $w_{\mathcal{L}}X$ . With this topology,  $w_{\mathcal{L}}X$  is compact [18, 4.4(d)]. Given  $x \in X$ , set  $\mathcal{U}_x := \{A \in \mathcal{L} \mid x \in A\}$ . Then  $\mathcal{U}_x \in w_{\mathcal{L}}X$ , and the map

$$\begin{aligned} X &\longrightarrow w_{\mathcal{L}}X \\ x \in X &\longmapsto \mathcal{U}_x \in w_{\mathcal{L}}X \end{aligned} \tag{6}$$

is a dense embedding, called the *Wallman compactification* of  $X$  induced by  $\mathcal{L}$ .

For any zero-dimensional Hausdorff space  $X$ , we note that any collection of clopen sets  $\mathcal{B}$  of  $X$  which also forms a closed base for the space, is, almost trivially, a Wallman base for the space, and that the generated Wallman compactification  $w_{\mathcal{B}}X$  is a zero-dimensional Hausdorff space [18, 4.7(b)].

Furthermore, we note that in the case where  $X$  is zero-dimensional and  $\mathcal{B}$  is a Wallman base for  $X$  consisting of clopen sets, the (bounded) functions from the uniformly closed subring of  $C(X)$  generated by the characteristic functions for the Boolean algebra of clopen sets generated by  $\mathcal{B}$  are exactly the functions which extend to  $w_{\mathcal{B}}X$  [18, 4I & 4J].



We next identify  $G$  with its Yosida representation  $\widehat{G} \subseteq C(X_G)$ , as given by Theorem 2.9. Recall the map  $\lambda: Z_G \rightarrow X_G$  as in (1–2). If  $\widehat{g} \in \widehat{G}$ , the assignment

$$\widehat{g} \in \widehat{G} \xmapsto{\mu} \widehat{g} \circ \lambda \in C(Z_G) \quad (7)$$

yields a unital homomorphism of  $\ell$ -groups  $\mu: \widehat{G} \rightarrow C(Z_G)$ , and a straightforward computation confirms that  $\mu$  is injective because  $\lambda$  is surjective. We therefore obtain a representation of  $G$  as

$$\mu(\widehat{G}) \subseteq C(Z_G). \quad (8)$$

**Remark 3.4.** We notice that the Yosida representation  $\widehat{g}$  of  $g$  is such that, for every  $\mathfrak{m} \in X_G$ ,  $\widehat{g}(\mathfrak{m}) = 0$  if, and only if,  $g \in \mathfrak{m}$ . This ensures the inclusion of zero sets

$$\mathbb{V}_m(g) \subseteq \{\mathfrak{p} \in Z_G \mid \mu(\widehat{g})(\mathfrak{p}) = \widehat{g}(\lambda(\mathfrak{p})) = 0\}.$$

Hence

$$\lambda^{-1}(\mathbb{S}_M(g)) = \mathbb{S}(\mu(\widehat{g})) \subseteq \mathbb{S}_m(g),$$

where  $\mathbb{S}(\mu(\widehat{g})) := Z_G \setminus \mu(\widehat{g})^{-1}(0)$  is the support of  $\mu(\widehat{g})$  at  $Z_G$ , and  $\lambda^{-1}(\mathbb{S}_M(g))$  is the preimage of  $\mathbb{S}_M(g)$  under  $\lambda$ .

We consider now the Boolean algebra  $\mathcal{B}(Z_G)$  of subsets of  $Z_G$  generated by the collection  $\{\mathbb{V}_m(g)\}_{g \in G}$ . By [3, 10.2.1], for every  $g \in G$ ,  $\mathbb{V}_m(g) \subseteq Z_g$  is a clopen set, hence all the elements of  $\mathcal{B}(Z_G)$  are clopens of  $Z_G$ . Moreover, since  $\{\mathbb{V}_m(g)\}_{g \in G} \subseteq \mathcal{B}(Z_G)$ , they also form a closed base for  $Z_G$ , and hence  $\mathcal{B}(Z_G)$  is a Wallman base of  $Z_G$ . We denote by

$$w_G Z_G \quad (9)$$

the Wallman compactification  $w_{\mathcal{B}(Z_G)} Z_G$  given by  $\mathcal{B}(Z_G)$ . The Boolean algebra of all clopens of  $w_G Z_G$ , written  $\text{Cp}(w_G Z_G)$ , can be identified with the elements of  $\mathcal{B}(Z_G)$ :

$$\mathcal{B}(Z_G) \cong \text{Cp}(w_G Z_G). \quad (10)$$

Let  $K(Z_G) \subseteq C(Z_G)$  be the Boolean algebra of characteristic functions for the elements of  $\mathcal{B}(Z_G)$ . Since each member of  $K(Z_G)$  extends uniquely to a member of the Boolean algebra  $K(w_G Z_G)$  of all characteristic functions in  $C(w_G Z_G)$  by [18, 4G], we have the isomorphism of Boolean algebras

$$K(Z_G) \cong K(w_G Z_G). \quad (11)$$

In summary, any two Boolean algebras in (10) and (11) are isomorphic, and their dual Stone space is (9).

**Remark 3.5.** It may happen that  $w_G Z_G$  is strictly smaller than the *Banaschewski compactification* of  $Z_G$ , the largest zero-dimensional compactification usually denoted  $\beta_0 Z_G$ . See Section 6.

**Lemma 3.6.** *With reference to the embedding (8), the uniform completion of the linear subspace of  $C(Z_G)$  generated by  $K(Z_G)$  contains  $\mu(\widehat{G})$ .*

*Proof.* Let  $V$  be the Riesz space generated by  $\mu(\widehat{G}) \cup K(Z_G)$  in  $C(Z_G)$ .

**Claim 3.7.** *The function  $1_{Z_G}$  is a strong unit for  $V$ .*

*Proof.* For each  $g \in G$  the function  $\widehat{g}$  is bounded in  $X_G$ . Since the image of the function  $\widehat{g} \circ \lambda$  coincides with the image of  $\widehat{g}$ , each element  $\mu(\widehat{g}) \in \mu(\widehat{G})$  is bounded. The characteristic functions in  $K(Z_G)$  are bounded functions by definition. Therefore each element of  $V$  is bounded, because the Riesz space operations preserve the property of being bounded. Hence  $1_{Z_G}$  is a strong unit for  $V$ .  $\square$

For each  $g \in G$  let  $\chi_g \in K(Z_G)$  be the characteristic function of  $\mathbb{S}_m(g)$ .

**Claim 3.8.**  $\chi_g^\perp = \mu(\widehat{g})^\perp$  in  $V$ .

*Proof of Claim 3.8.* Let  $v \in V$  such that  $v \in \chi_g^\perp$ . Hence  $v(\mathbf{p}) \wedge \chi_g(\mathbf{p}) = 0$  for every  $\mathbf{p} \in Z_G$ . As a consequence, if  $\chi_g(\mathbf{p}) \neq 0$ , then  $v(\mathbf{p}) = 0$ . Therefore, whenever  $\mu(\widehat{g})(\mathbf{p}) \neq 0$ , we have  $v(\mathbf{p}) = 0$  by Remark 3.4. Then  $v(\mathbf{p}) \wedge \mu(\widehat{g})(\mathbf{p}) = 0$  for all  $\mathbf{p} \in Z_G$ , whence  $v \in \mu(\widehat{g})^\perp$ . This proves the inclusion  $\chi_g^\perp \subseteq \mu(\widehat{g})^\perp$ .

For the other inclusion, suppose there exists  $v \in V$  such that  $v \in \mu(\widehat{g})^\perp \setminus \chi_g^\perp$ . Then the supports  $\mathbb{S}(v)$  and  $\mathbb{S}(\mu(\widehat{g}))$  are disjoint, and there is  $\mathbf{p} \in Z_G$  such that  $|v(\mathbf{p})| \wedge |\chi_g(\mathbf{p})| \neq 0$ . By definition of  $\chi_g$ ,  $\chi_g^{-1}(1) = \mathbb{S}(\chi_g) = \mathbb{S}_m(g)$  and therefore  $\mathbf{p}$  is in  $\mathbb{S}(v) \cap \mathbb{S}_m(g)$ , which is open since  $v$  is a continuous function on  $Z_G$  and  $\mathbb{S}_m(g)$  is a basic open subset of  $Z_G$ . Hence there exists a (nonempty) basic open set  $\mathbb{S}_m(h)$  (with  $0 \neq h \in G$ ) such that  $\mathbf{p} \in \mathbb{S}_m(h) \subseteq \mathbb{S}(v) \cap \mathbb{S}_m(g)$ . Let  $\mathbf{m} \in \mathbb{S}_M(h)$ . Then  $h \notin \mathbf{m}$ , whence  $h \notin \mathbf{q}$  for every  $\mathbf{q} \in \lambda^{-1}(\mathbf{m})$  and  $\lambda^{-1}(\mathbf{m}) \subseteq \mathbb{S}_m(h)$ . Since  $\mathbb{S}_m(h)$  is disjoint from  $\mathbb{S}(\mu(\widehat{g}))$ , and  $\mathbb{S}(\mu(\widehat{g})) = \lambda^{-1}(\mathbb{S}_M(g))$  by Remark 3.4,  $\mathbf{q} \notin \lambda^{-1}(\mathbb{S}_M(g))$  for every  $\mathbf{q} \in \lambda^{-1}(\mathbf{m})$ . Then  $\mathbf{m} \in \mathbb{V}_M(g)$  and  $g \in \mathbf{m}$ . This ensures  $\mathbb{S}_M(h) \subseteq \mathbb{V}_M(g)$ , and hence  $|h| \wedge |g| = 0$ . By primality of  $\mathbf{p}$ , either  $h \in \mathbf{p}$  or  $g \in \mathbf{p}$ , in contradiction with  $\mathbf{p} \in \mathbb{S}_m(h) \subseteq \mathbb{S}_m(g)$ . This ensures  $v \in \chi_g^\perp$ , whence  $\mu(\widehat{g})^\perp \subseteq \chi_g^\perp$ .  $\square$

**Claim 3.9.**  $V$  is projectable.

*Proof of Claim 3.9.* We need to show that each  $v \in V$  induces a product splitting  $V \cong v^{\perp\perp} \times v^\perp$ . By Claim 3.8,  $\chi_g^{\perp\perp} = \mu(\widehat{g})^{\perp\perp}$  for all  $g \in G$ . But since  $\chi_g$  is a component of  $1_{Z_G}$  (because  $\chi_g \vee (1_{Z_G} - \chi_g) = 1_{Z_G}$  and  $\chi_g \wedge (1_{Z_G} - \chi_g) = 0$ ), there is an induced splitting  $V \cong \chi_g^{\perp\perp} \times \chi_g^\perp$ . This shows that each element of  $V$  that lies in the generating set  $\mu(\widehat{G}) \cup K(Z_G)$  induces the splitting above. A routine induction argument now shows that the splitting property is preserved by sums, meets and joins, and products by real scalars, thus settling the claim.  $\square$

By the preceding claim, we may apply Freudenthal's Spectral Theorem 2.5 to  $(V, 1_{Z_G})$ , and infer that each element of  $\mu(\widehat{G})$  is a  $1_{Z_G}$ -uniform limit of a sequence of elements in the linear subspace of  $C(Z_G)$  generated by  $K(Z_G)$ . Since the norm induced by  $1_{Z_G}$  on  $C(Z_G)$  coincides with the supremum norm, this completes the proof.  $\square$

**Lemma 3.10.** For each  $g \in G$ , there exists a unique continuous extension of  $\mu(\widehat{g}) \in C(Z_G)$  to an element  $g^\sharp \in C(w_G Z_G)$ . That is,  $g^\sharp$  is the unique such element whose restriction to  $Z_G$  is  $\mu(\widehat{g})$ . In symbols,

$$g^\sharp|_{Z_G} = \mu(\widehat{g}). \quad (12)$$

*Proof.* Indeed, by Lemma 3.6 there is a sequence  $\{c_i\}_{i \geq 1}$  of linear combinations of elements of  $K(Z_G)$  that converges uniformly to  $\mu(\widehat{g})$ . Each member of  $K(Z_G)$  extends uniquely to a member of  $K(w_G Z_G)$  [18, 4G], and therefore each  $c_i$  extends to a linear combination  $k_i$  of elements of  $K(w_G Z_G)$ . It is now elementary to check

that  $\{k_i\}_{i \geq 1}$  is a Cauchy sequence in  $C(w_G Z_G)$  because  $\{c_i\}_{i \geq 1}$  is one in  $C(Z_G)$ . Take  $g^\sharp$  to be the limit of  $\{k_i\}_{i \geq 1}$ , which is of course a continuous function by the Uniform Limit Theorem. Finally, note that  $g^\sharp$  has property (12) by construction, and is the unique member of  $C(w_G Z_G)$  with this property because  $Z_G$  is dense in its  $w_G$ -compactification, and the codomain of the functions — namely,  $\mathbb{R}$  — is Hausdorff.  $\square$

In light of Lemma 3.10, the function

$$.^\sharp: G \hookrightarrow C(w_G Z_G) \quad (13)$$

that acts by  $g \mapsto g^\sharp$  is injective. It is elementary that this embedding preserves the lattice and group structure of  $G$ , and is also unit-preserving. We have therefore proved:

**Theorem 3.11.** *Each  $\mathbf{U}$ -object  $(G, u)$  has a representation into  $C(w_G Z_G)$  as in (13).  $\square$*

**Definition 3.12.** We write  $\mathcal{P}(G)$  for the  $\ell$ -subgroup of  $C(w_G Z_G)$  generated by

$$G^\sharp \cup K(w_G Z_G). \quad (14)$$

We further write

$$\pi: G \hookrightarrow \mathcal{P}(G) \quad (15)$$

for the  $\mathbf{U}$ -monomorphism of  $G$  into  $\mathcal{P}(G)$  obtained by restricting the codomain of (13) to  $\mathcal{P}(G)$ .

#### 4. CHARACTERISATION OF THE ELEMENTS OF $\mathcal{P}(G)$

In this section we characterise the functions in  $C(w_G Z_G)$  that lie in  $\mathcal{P}(G)$ . We begin by preparing two lemmas.

**Lemma 4.1.** *There is a homeomorphism  $\text{Max } \mathcal{P}(G) \cong w_G Z_G$ .*

*Proof.* Indeed, the characteristic functions  $K(w_G Z_G) \subseteq \mathcal{P}(G)$  separate the points of  $w_G Z_G$ , because the latter is zero-dimensional; now apply Yosida's Theorem 2.9.  $\square$

**Remark 4.2.** The next lemma is a consequence of more general results, cf. [14, Proposition 2.4]. We provide a proof here, for the reader's convenience.

**Lemma 4.3.** *Let  $g \in G$ , and let  $\chi \in G$  be a component of the unit  $u$ . Let us identify  $G$  with its Yosida representation  $\widehat{G} \subseteq C(X_G)$ . The pointwise product  $g\chi$  defined by  $(g\chi)(x) = g(x)\chi(x)$  for each  $x \in X_G$  is a continuous function, and hence an element of  $C(X_G)$ . Then  $g\chi \in \widehat{G}$ .*

*Proof.* (Skipping all trivialities, in this proof we identify isomorphism with equality without further warning.) Since  $\chi$  is a component of  $u$  we have a product splitting  $\widehat{G} = \chi^\perp \times \chi^{\perp\perp}$  ( $\perp$  computed in  $\widehat{G}$ ), and a corresponding disjoint union decomposition  $X_G = A \sqcup B$ ,  $A := \chi^{-1}(0)$ ,  $B := \chi^{-1}(1)$ ,  $A$  and  $B$  disjoint clopens in  $X_G$ . Then, clearly,  $C(X_G) = C(A) \times C(B)$ ,  $\chi^\perp \subseteq C(A)$ , and  $\chi^{\perp\perp} \subseteq C(B)$ . Now since  $g \in \chi^\perp \times \chi^{\perp\perp}$ ,  $g$  may be uniquely expressed as a sum  $g_1 + g_2$ ,  $g_i \in \widehat{G}$ ,  $g_1 \in \chi^\perp$ ,  $g_2 \in \chi^{\perp\perp}$ . Then  $g$  and  $g_2$  agree over  $B$ , so that  $g\chi = g_2\chi = g_2 \in \widehat{G}$ , and the lemma is proved.  $\square$

**Remark 4.4.** Let  $0 \leq g \in G$ , and let  $\chi \in G$  be a component of the unit  $u$ . Identifying  $G$  with its Yosida representation  $\widehat{G} \subseteq C(X_G)$ , we notice that the function  $g$  is bounded on the support of the characteristic function  $\chi$ . Therefore, there exists a (unique minimal) integer  $n \geq 0$  such that  $g \leq n\chi$  holds on the support of  $\chi$ , and hence  $g\chi = g \wedge n\chi$  holds in  $G$ . This yields an explicit representation of the product  $g\chi$  discussed in Lemma 4.3, using only the operations of  $G$ . Any element  $g \in G$ , indeed, can be written as the difference  $g^+ - g^-$  between its *positive part*  $g^+ := g \vee 0$  and its *negative part*  $g^- := (-g) \vee 0$ , with  $0 \leq g^+, g^- \in G$ . As a consequence, there exist two (unique minimal) integers  $n_+, n_- \geq 0$  such that  $g\chi = (g^+ \wedge n_+\chi) - (g^- \wedge n_-\chi)$ .

In the following, we use the product  $g\chi$  for brevity, but each such occurrence may be replaced by the equivalent expression  $(g^+ \wedge n_+\chi) - (g^- \wedge n_-\chi)$ .

By a *partition of unity* in a  $\mathbf{U}$ -object  $(G, u)$  we mean in this paper a finite family of non-zero elements  $P := \{\chi_i\}_{i=1}^l$  of  $G$  such that  $\sum_{i=1}^l \chi_i = u$ , and  $\chi_i \wedge \chi_j = 0$  whenever  $i \neq j$ . It is elementary that each  $\chi_i$  is a component of  $u$ . It follows that, in the Yosida representation  $\widehat{G}$  of  $G$ , each  $\widehat{\chi}_i$  is a characteristic function.

We can now prove:

**Theorem 4.5.** *For each  $e \in C(w_G Z_G)$ , the following are equivalent.*

- (1)  $e \in \mathcal{P}(G)$ .
- (2) *There exists a partition of unity  $\chi_1, \dots, \chi_l$  in  $\mathcal{P}(G)$  — equivalently, in  $C(w_G Z_G)$  — along with elements  $a_1, \dots, a_l \in G$ , such that*

$$e = \sum_{i=1}^l a_i^\# \chi_i, \quad (16)$$

where  $\#$  is the embedding (13), and  $a_i^\# \chi_i$  denotes the pointwise product of  $a_i^\#$  and  $\chi_i$  in  $C(w_G Z_G)$ .

*Proof.* First, let us explicitly note that  $\mathcal{P}(G)$  and  $C(w_G Z_G)$  have the same collection of partitions of unity because  $K(w_G Z_G) \subseteq \mathcal{P}(G)$ .

(1) $\Rightarrow$ (2) Recall that  $\mathcal{P}(G)$  is the  $\ell$ -subgroup of  $C(w_G Z_G)$  generated by the set (14). Hence by the elementary theory of lattice-groups we can write  $e$  as

$$\bigwedge_{i \in I} \bigvee_{j \in J} (g_{ij}^\# + c_{ij} k_{ij}),$$

where  $I$  and  $J$  are finite sets of indices, at least one of which is non-empty,  $g_{ij} \in G$ ,  $c_{ij} \in \mathbb{Z}$  and  $k_{ij} \in K(w_G Z_G)$ . Now for each  $k_{ij}$ , we obtain associated clopen subsets of  $w_G Z_G$ , namely their supports and their complements. This (necessarily non-empty) collection of clopens obviously covers  $w_G Z_G$ . It is elementary that we can refine this cover into a finite partition  $\{D_m\}_{m \in M}$  of clopens of the space by taking intersections and set-theoretic differences.

On each  $D_m$ , each  $k_{ij}$  is constant — either zero or one — by construction. Let us define the element  $\delta_{ij}^m \in G$  by setting

$$\delta_{ij}^m := \begin{cases} u & \text{if } D_m \subseteq k_{ij}^{-1}(1), \\ 0 & \text{if } D_m \subseteq k_{ij}^{-1}(0). \end{cases} \quad (17)$$

Now consider the element of  $\mathcal{P}(G)$

$$e_m := \bigwedge_{i \in I} \bigvee_{j \in J} \left( g_{ij}^\# + c_{ij}(\delta_{ij}^m)^\# \right).$$

Observe that *the function  $e_m$  agrees over  $D_m$  with the function  $e$ , for each  $m \in M$ . This follows immediately from our definition of  $\delta_{ij}^m$  in (17) above. Moreover, since  $\#$  is an  $\ell$ -homomorphism, we have*

$$e_m = \left( \bigwedge_{i \in I} \bigvee_{j \in J} (g_{ij} + c_{ij}\delta_{ij}^m) \right)^\# = a_m^\#,$$

where  $a_m := \bigwedge_{i \in I} \bigvee_{j \in J} (g_{ij} + c_{ij}\delta_{ij}^m) \in G$ . Hence, if we let  $\chi_m$  be the characteristic function of  $D_m$ , we conclude

$$e = \sum_{m \in M} a_m^\# \chi_m,$$

as was to be shown.

(2) $\Rightarrow$ (1) This follows at once from Lemmas 4.1 and 4.3. □

## 5. CONSTRUCTION OF THE PROJECTABLE HULL

Our final aim is to show that the embedding (13) provides a description of the projectable hull of  $G$ . This is our main result:

**Theorem 5.1.** *For any  $\mathbf{U}$ -object  $(G, u)$ , the embedding  $\pi: G \hookrightarrow \mathcal{P}(G)$  as in (15) is the projectable hull of  $G$ .*

*Proof.* The proof that  $\mathcal{P}(G)$  is projectable is identical to that of Claim 3.9.

To prove that the map  $\pi$  is an essential extension, we verify (3) in Lemma 2.6. Pick  $0 < e \in \mathcal{P}(G)$ , and express it as  $e = \sum_{i=1}^l a_i^\# \chi_i$  by Theorem 4.5, for a partition of unity  $\{\chi_i\}_{i=1}^l$  in  $\mathcal{P}(G)$  and elements  $\{a_i\}_{i=1}^l$  in  $G$ . Since  $e > 0$ , we must have  $a_i^\# \chi_i \geq 0$  for each  $i$ , and  $a_{i_0}^\# \chi_{i_0} > 0$  for some  $i_0$ . It is enough to show that there is  $h \in G$  such that  $0 < h^\# \leq n a_{i_0}^\# \chi_{i_0}$ , for then  $0 < h^\# \leq n e$  follows easily. By (11), we can identify the characteristic functions in  $K(w_G Z_G)$  with the elements of  $K(Z_G)$ . Set  $a := a_{i_0}$  and  $\chi := \chi_{i_0} \wedge \chi_a$ , where  $\chi_a \in K(Z_G)$  is the characteristic function of  $\mathbb{S}_m(a)$ . Since the support  $\mathbb{S}(\chi) = \chi^{-1}(1)$  is clopen and nonempty (because  $a^\# \chi \neq 0$ ), there exists  $0 < g \in G$  such that the basic open set  $\mathbb{S}_m(g)$  is nonempty and contained in  $\mathbb{S}(\chi) \subseteq \mathbb{S}_m(a)$ . Hence  $h := g \wedge a > 0$ , and  $\mathbb{S}(h^\#) \subseteq \mathbb{S}_m(h) \subseteq \mathbb{S}(\chi)$ . We now claim that  $h^\# \leq a^\# \chi$ . It is enough to prove that the inequality holds for a point  $x \in w_G Z_G$  in the support of  $\chi$ , where  $\chi(x) = 1$ . Since  $\#$  is an  $\ell$ -homomorphism,  $h^\# = a^\# \wedge g^\# > 0$ . If  $a^\#(x) \leq g^\#(x)$ , then  $h^\#(x) = a^\#(x)$ , and the inequality holds. Otherwise, we have  $h^\#(x) = g^\#(x) < a^\#(x)$ . This settles the claim and completes the proof that  $\pi$  is essential.

To show  $\mathcal{P}(G)$  is a hull, it suffices to show that given the (unital) essential embedding  $\iota$  into  $H$  there exists an (automatically essential and unital) embedding

$\varphi$  making the diagram below commute:

$$\begin{array}{ccc} (G, u) & \xleftarrow{\pi} & (\mathcal{P}(G), u^\sharp) \\ & \searrow \iota & \downarrow \varphi \\ & & (H, v) \end{array} \quad (*)$$

We define a function  $\varphi: \mathcal{P}(G) \rightarrow H$  as follows. First we set

$$\varphi(a^\sharp) := \iota(a), \text{ for each } a \in G. \quad (18)$$

Since  $H$  is projectable, for each  $g \in G$  there is a unique component  $\sigma_g \in H$  of the unit  $v$  that satisfies  $\iota(g)^{\perp\perp} = \sigma_g^{\perp\perp}$ , where  $\perp$  is computed in  $H$ . By (11), we identify the components of the unit  $u^\sharp$  in  $\mathcal{P}(G)$  with the characteristic functions in  $K(Z_G)$ . Let  $\chi_g \in \mathcal{P}(G)$  be the characteristic function of  $\mathbb{S}_m(g)$  for  $g \in G$ . The analogous result of Claim 3.8 for  $\mathcal{P}(G)$  ensures that  $\chi_g^\perp = \pi(g)^\perp$  in  $\mathcal{P}(G)$ . We set

$$\varphi(\chi_g) := \sigma_g. \quad (19)$$

Let  $K(H)$  be the Boolean algebra of components of the unit  $v$  in  $H$ .

**Claim 5.2.** *The map  $\varphi: \{\chi_g\}_{g \in G} \rightarrow K(H)$  defined as in (19) is a lattice homomorphism.*

*Proof.* Since  $\iota$  is a unital homomorphism, trivially  $\varphi(\chi_0) = 0$  and  $\varphi(\chi_u) = v$ . Let  $g_1, g_2 \in G$ . It is easy to check that  $\chi_{g_1} \wedge \chi_{g_2} = \chi_{g_1 \wedge g_2}$  and  $\chi_{g_1} \vee \chi_{g_2} = \chi_{g_1 \vee g_2}$ . Let  $\varphi(\chi_{g_1}) = \sigma_{g_1}$ ,  $\varphi(\chi_{g_2}) = \sigma_{g_2}$ ,  $\varphi(\chi_{g_1 \wedge g_2}) = \sigma_{g_1 \wedge g_2}$  and  $\varphi(\chi_{g_1 \vee g_2}) = \sigma_{g_1 \vee g_2}$  as in (19). Hence, by (4) and (5),

$$\begin{aligned} \sigma_{g_1 \wedge g_2}^{\perp\perp} &= \iota(g_1 \wedge g_2)^{\perp\perp} = (\iota(g_1) \wedge \iota(g_2))^{\perp\perp} = \\ &= \iota(g_1)^{\perp\perp} \cap \iota(g_2)^{\perp\perp} = \sigma_{g_1}^{\perp\perp} \cap \sigma_{g_2}^{\perp\perp} = (\sigma_{g_1} \wedge \sigma_{g_2})^{\perp\perp}, \end{aligned}$$

and

$$\begin{aligned} \sigma_{g_1 \vee g_2}^{\perp\perp} &= \iota(g_1 \vee g_2)^{\perp\perp} = (\iota(g_1) \vee \iota(g_2))^{\perp\perp} = \\ &= (\iota(g_1)^{\perp\perp} \cup \iota(g_2)^{\perp\perp})^{\perp\perp} = (\sigma_{g_1}^{\perp\perp} \cup \sigma_{g_2}^{\perp\perp})^{\perp\perp} = (\sigma_{g_1} \vee \sigma_{g_2})^{\perp\perp}. \end{aligned}$$

This ensures  $\sigma_{g_1 \wedge g_2} = \sigma_{g_1} \wedge \sigma_{g_2}$  and  $\sigma_{g_1 \vee g_2} = \sigma_{g_1} \vee \sigma_{g_2}$ . Therefore  $\varphi(\chi_{g_1} \wedge \chi_{g_2}) = \varphi(\chi_{g_1 \wedge g_2}) = \varphi(\chi_{g_1}) \wedge \varphi(\chi_{g_2})$  and  $\varphi(\chi_{g_1} \vee \chi_{g_2}) = \varphi(\chi_{g_1 \vee g_2}) = \varphi(\chi_{g_1}) \vee \varphi(\chi_{g_2})$ , and the claim is proved.  $\square$

The previous Claim 5.2, together with [1, Lemma 1 in Section V.4], ensures that  $\varphi$  extends uniquely to a homomorphism  $\varphi: K(Z_G) \rightarrow K(H)$  of Boolean algebras. Specifically, since the Boolean algebra  $K(Z_G)$  can be generated by the set  $\{\chi_g\}_{g \in G}$  using the negation  $u^\sharp -$  and the meet operation  $\wedge$ , the image of a general element of  $K(Z_G)$  under  $\varphi$  can be inductively described as follows. Let  $\chi_1, \chi_2, \alpha, \beta_1, \beta_2 \in K(Z_G)$  such that  $\chi_1 = u^\sharp - \alpha$  and  $\chi_2 = \beta_1 \wedge \beta_2$ . Then

$$\varphi(\chi_1) = v - \varphi(\alpha) \quad \text{and} \quad \varphi(\chi_2) = \varphi(\beta_1) \wedge \varphi(\beta_2). \quad (20)$$

Finally, for a general  $e \in \mathcal{P}(G)$ , we first write  $e$  as in (16) using Theorem 4.5, and then, using (18–20), we set

$$\varphi(e) := \sum_{i=1}^l \varphi(a_i^\sharp) \varphi(\chi_i). \quad (21)$$

Since each product  $\varphi(a_i^\sharp)\varphi(\chi_i)$  is an element of  $H$  by Lemma 4.3,  $\varphi(e)$  as in (21) is an element of  $H$ .

We next verify that  $\varphi$  is a well-defined function. Given a decomposition (16) of  $e \in \mathcal{P}(G)$  as in Theorem 4.5, suppose  $e = \sum_{j=1}^t b_j^\sharp \xi_j$  is another such decomposition. It suffices to show that

$$\sum_{i=1}^l \varphi(a_i^\sharp)\varphi(\chi_i) = \sum_{j=1}^t \varphi(b_j^\sharp)\varphi(\xi_j). \quad (22)$$

It is elementary to verify that the set  $\{\chi_i \wedge \xi_j \mid \chi_i \wedge \xi_j \neq 0\}$  forms a partition of unity that refines both  $\{\chi_i\}_{i=1}^l$  and  $\{\xi_j\}_{j=1}^t$ ; that is, each  $\chi_i$  and  $\xi_j$  is a sum (or join, by pairwise disjointness) of elements  $\chi_{i'} \wedge \xi_{j'} \neq 0$ . It follows that  $e$  can be expressed in two ways as

$$e = \sum a_i^\sharp(\chi_i \wedge \xi_j) = \sum b_j^\sharp(\chi_i \wedge \xi_j),$$

and hence  $a_i^\sharp(\chi_i \wedge \xi_j) = b_j^\sharp(\chi_i \wedge \xi_j)$  for all  $i, j$ .

In the simplest case, we must prove:

**Claim 5.3.** *Assuming  $\chi \in K(Z_G)$ , suppose  $a^\sharp\chi = b^\sharp\chi$ . Then  $\varphi(a^\sharp\chi) = \iota(a)\varphi(\chi) = \iota(b)\varphi(\chi) = \varphi(b^\sharp\chi)$ .*

*Proof.* Let  $\nu_{\mathcal{P}(G)}: \text{Pol } \mathcal{P}(G) \rightarrow \text{Pol } \pi(G)$  and  $\nu_H^{-1}: \text{Pol } \iota(G) \rightarrow \text{Pol } H$  be the isomorphisms of Boolean algebras given in Lemma 2.6. Then also  $\nu_\pi := \pi^{-1} \circ \nu_{\mathcal{P}(G)}: \text{Pol } \mathcal{P}(G) \rightarrow \text{Pol } G$  and  $\nu_\iota := \nu_H^{-1} \circ \iota: \text{Pol } G \rightarrow \text{Pol } H$  are isomorphisms of Boolean algebras. Since  $\chi^{\perp \mathcal{P}(G)}$  is a polar, there exists a unique polar  $P_\chi \in \text{Pol } G$  such that  $P_\chi^{\perp G} = \nu_\pi(\chi^{\perp \mathcal{P}(G)})$ . From  $a^\sharp\chi = b^\sharp\chi$ , we have  $\pi(a - b) = (a^\sharp - b^\sharp) \in \chi^{\perp \mathcal{P}(G)}$ . Hence  $a - b \in P_\chi^{\perp G}$  and  $\iota(a) - \iota(b) = \iota(a - b) \in \iota(P_\chi)^{\perp \iota(G)} = \iota(P_\chi)^{\perp H} \cap \iota(G)$ . To complete the proof we show  $\iota(P_\chi)^{\perp H} = \varphi(\chi)^{\perp H}$ , whence  $(\iota(a) - \iota(b)) \in \varphi(\chi)^{\perp H}$  and  $\iota(a)\varphi(\chi) = \iota(b)\varphi(\chi)$ .

We proceed by induction on the structure of  $\chi$  using (20). In the basic case we suppose  $\chi := \chi_g$ , for  $g \in G$ . Then  $\chi_g^{\perp \mathcal{P}(G)} = \pi(g)^{\perp \mathcal{P}(G)}$  by Claim 3.8, whence  $P_\chi^{\perp G} = \nu_\pi(\pi(g)^{\perp \mathcal{P}(G)}) = \pi^{-1}(\nu_{\mathcal{P}(G)}(\pi(g)^{\perp \mathcal{P}(G)})) = \pi^{-1}(\pi(g)^{\perp \mathcal{P}(G)} \cap \pi(G)) = g^{\perp G} = (g^{\perp \perp G})^{\perp G}$ , and  $P_\chi = g^{\perp \perp G}$ . By definition of  $\varphi$  on  $\{\chi_g\}_{g \in G}$ , we have  $\varphi(\chi)^{\perp H} = \iota(g)^{\perp H} = \iota(g^{\perp \perp G})^{\perp H}$ , where the last equality is a direct consequence of the essentiality of  $\iota$  (see [7, §2]).

We now consider the case  $\chi := u^\sharp - \alpha$ , and suppose  $\iota(P_\alpha)^{\perp H} = \varphi(\alpha)^{\perp H}$ . Since  $\chi$  and  $\alpha$  are components of unity,  $\chi^{\perp \mathcal{P}(G)} = \alpha^{\perp \perp \mathcal{P}(G)}$ , whence  $P_\chi^{\perp G} = P_\alpha^{\perp \perp G}$  and  $\iota(P_\chi)^{\perp H} = \iota(P_\alpha)^{\perp \perp H}$ , because both  $\nu_\pi$  and  $\nu_\iota$  are isomorphisms of Boolean algebras. Since also  $\varphi: K(Z_G) \rightarrow K(H)$  is an isomorphism of Boolean algebras,  $\varphi(\chi) = v - \varphi(\alpha)$ . Therefore,  $\iota(P_\chi)^{\perp H} = \iota(P_\alpha)^{\perp \perp H} = \varphi(\alpha)^{\perp \perp H} = \varphi(\chi)^{\perp H}$ .

The case  $\chi := \beta_1 \wedge \beta_2$  is proved in a similar way. Suppose  $\iota(P_{\beta_1})^{\perp H} = \varphi(\beta_1)^{\perp H}$  and  $\iota(P_{\beta_2})^{\perp H} = \varphi(\beta_2)^{\perp H}$ . By (4),  $\chi^{\perp \perp \mathcal{P}(H)} = (\beta_1 \wedge \beta_2)^{\perp \perp \mathcal{P}(H)} = \beta_1^{\perp \perp \mathcal{P}(H)} \cap \beta_2^{\perp \perp \mathcal{P}(H)}$ , whence  $\chi^{\perp \mathcal{P}(H)} = (\beta_1^{\perp \perp \mathcal{P}(H)} \cap \beta_2^{\perp \perp \mathcal{P}(H)})^{\perp \mathcal{P}(H)} = \beta_1^{\perp \mathcal{P}(H)} \vee \beta_2^{\perp \mathcal{P}(H)}$ . As a consequence,  $P_\chi^{\perp \mathcal{P}(H)} = P_{\beta_1}^{\perp \mathcal{P}(H)} \vee P_{\beta_2}^{\perp \mathcal{P}(H)}$  and  $\iota(P_\chi)^{\perp H} = \iota(P_{\beta_1})^{\perp H} \vee \iota(P_{\beta_2})^{\perp H}$ . Since  $\varphi(\chi) = \varphi(\beta_1) \wedge \varphi(\beta_2)$ , we have  $\varphi(\chi)^{\perp \perp \mathcal{P}(H)} = \varphi(\beta_1)^{\perp \perp \mathcal{P}(H)} \cap \varphi(\beta_2)^{\perp \perp \mathcal{P}(H)}$ , whence  $\varphi(\chi)^{\perp \mathcal{P}(H)} = \varphi(\beta_1)^{\perp \mathcal{P}(H)} \vee \varphi(\beta_2)^{\perp \mathcal{P}(H)} = \iota(P_{\beta_1})^{\perp H} \vee \iota(P_{\beta_2})^{\perp H} = \iota(P_\chi)^{\perp H}$ . The claim is settled.  $\square$

Since  $\varphi: K(Z_G) \rightarrow K(H)$  is a homomorphism of Boolean algebras,  $\varphi$  preserves partitions of unity and  $\varphi(\chi \wedge \xi) = \varphi(\chi) \wedge \varphi(\xi)$ . Therefore

$$\begin{aligned} \sum \varphi(a_i^\#) \varphi(\chi_i \wedge \xi_j) &= \sum \varphi(b_j^\#) \varphi(\chi_i \wedge \xi_j), \\ \sum_j \varphi(a_i^\#) \varphi(\chi_i \wedge \xi_j) &= \varphi(a_i^\#) \varphi(\chi_i) \text{ for all } i\text{'s}, \\ \sum_i \varphi(b_j^\#) \varphi(\chi_i \wedge \xi_j) &= \varphi(b_j^\#) \varphi(\xi_j) \text{ for all } j\text{'s}. \end{aligned}$$

These together prove (22).

The map  $\varphi$  makes the diagram (\*) commute by construction. To show that it is an  $\ell$ -homomorphism one argues as follows. Given  $e + f \in \mathcal{P}(G)$ , to prove  $\varphi(e + f) = \varphi(e) + \varphi(f)$ , we first take decompositions of  $e + f$ ,  $e$  and  $f$  as in (16) of Theorem 4.5. We then pick a joint refinement of the three partitions of unity involved. We finally proceed as in the preceding argument that shows  $\varphi$  is well-defined. We omit the elementary details. The argument for the remaining operations is analogous.

To show  $\varphi$  is injective, consider  $e \neq f \in \mathcal{P}(G)$ . Using again decompositions as in (16) of Theorem 4.5, and a common refinement of the associated partitions, we see that  $e$  and  $f$  must differ on some element of the common refinement. Injectivity of  $\varphi$  then follows at once from the injectivity of  $\iota$  and  $\pi$ .  $\square$

## 6. EXAMPLES

We close the paper with two examples. Example 6.1 shows that the space  $Z_G$  of minimal primes can fail to be compact even though its  $G$ -indexed compactification  $w_G Z_G$  is actually  $\beta_0 Z_G$ , the largest zero-dimensional compactification of  $Z_G$ . This happens when the base of clopens of  $Z_G$  indexed by elements of  $G$  is not a Boolean algebra — because it fails to be closed under complements — and yet it is large enough that the Boolean algebra it generates consists of all clopens of  $Z_G$ . We note in passing that the  $\ell$ -group in this first example can be shown to be finitely generated. Example 6.2, by contrast, is an instance where the  $G$ -indexed compactification  $w_G Z_G$  is strictly smaller than  $\beta_0 Z_G$ . Here  $G$  is of the form  $C(X)$  for  $X$  a compact Hausdorff space.

**Example 6.1.** Let  $\mathcal{M}$  be the set of continuous and piecewise linear functions  $f: [0, 1]^2 \rightarrow \mathbb{R}$  with integer coefficients. This means that  $f$  is continuous, and that there are finitely many triplets of integers  $(a_i, b_i, c_i) \in \mathbb{Z}^3$  such that for all  $(x, y) \in [0, 1]^2$  we have  $f(x, y) = a_i x + b_i y + c_i$  for some  $i$ . When equipped with the pointwise operations of minimum, maximum, and addition,  $\mathcal{M}$  is an  $\ell$ -group. The function identically equal to 1 is a strong unit.

Let

$$P := \{(x, y) \in [0, 1]^2 \mid y = x^2\} \cup \{(x, y) \in [0, 1]^2 \mid y = 0\},$$

and let  $(G, 1_P)$  be the unital  $\ell$ -group obtained by restricting each element of  $\mathcal{M}$  to  $P$ , where  $1_P: P \rightarrow \mathbb{R}$  is the function constantly equal to 1. We now show that  $Z_G := \text{Min } G$  fails to be compact. For this, consider the projection function  $\pi_y: (x, y) \mapsto y$ , along with the set of functions  $h_n$  ( $0 < n \in \mathbb{N}$ ) which are equal to  $nx - 1$  on the segment  $\{(x, y) \in [0, 1]^2 \mid y = 0 \text{ and } x \in [\frac{1}{n}, 1]\}$  and to 0 elsewhere. These are elements of  $G$ . Then  $\{\mathbb{S}_m(\pi_y)\} \cup \{\mathbb{S}_m(h_n)\}_{n>0}$  is an open cover of  $Z_G$  without a finite subcover. According to Lemma 3.2,  $G$  is not complemented. Indeed,



by piecewise linearity and continuity, there can be no  $f \in \mathcal{M}$  that restricts to  $P$  so as to satisfy  $\pi_y^{\perp\perp} = f^{\perp}$ . By the same lemma,  $\{\mathbb{V}_m(g)\}_{g \in G}$  is not a Boolean algebra. Indeed, it does not contain the complement  $\mathbb{S}_m(\pi_y)$  of  $\mathbb{V}_m(\pi_y)$ . In particular,  $\{\mathbb{V}_m(g)\}_{g \in G}$  is not the collection of all clopens of  $Z_G$ , because  $\mathbb{S}_m(\pi_y)$  is clopen — it is both a basic open set and the intersection of the closed sets in  $\{\mathbb{V}_m(h_n)\}_{n>0}$ . However, the Boolean algebra  $\mathcal{B}(Z_G)$  generated by  $\{\mathbb{V}_m(g)\}_{g \in G}$  is the collection of *all* clopens of  $Z_G$ , as can be seen by direct inspection. Therefore  $\text{Max } \mathcal{P}(G) \cong w_G Z_G$  is homeomorphic to  $\beta_0 Z_G$ , the maximal zero-dimensional compactification of  $Z_G$ .

**Example 6.2.** Let  $D$  be the discrete space with  $|D| = \aleph_1$ , let  $\alpha D = D \cup \{\alpha\}$  be the one-point compactification of  $D$ , and let  $G = C(\alpha D)$  so that  $\text{Max } G \cong \alpha D$ . Then  $Z_G := \text{Min } G$  can be identified with the collection of fixed ultrafilters on  $D$  together with the free ultrafilters on  $D$  which contain a countable subset. The basic closed sets of  $Z_G$ , of the form

$$\{\mathcal{U} \in Z_G \mid g^{-1}(0) \in \mathcal{U}\},$$

are actually clopen. We show that the Boolean algebra generated by these basic clopen sets is *not* the Boolean algebra of all clopens, but a proper subalgebra. For, let  $D_1$  and  $D_2$  be elements of a partition of  $D$  with  $|D_i| = \aleph_1$ ,  $i = 1, 2$ . Then we have an induced partition of  $Z_G$  into two clopens:

$$Z_G = \{\mathcal{U} \mid \exists C \in \mathcal{U} \text{ with } C \subseteq D_1\} \sqcup \{\mathcal{U} \mid \exists C \in \mathcal{U} \text{ with } C \subseteq D_2\}.$$

Neither summand in the partition above is a basic closed set of  $Z_G$ . Therefore  $\text{Max } \mathcal{P}(G) \cong w_G Z_G \not\cong \beta_0 Z_G$ .

**Remark 6.3.** In case  $G = C(X)$  for some compact Hausdorff space  $X$ , the main result of this paper is related to several others in the literature, and especially to the construction of zero-dimensional covers of  $X$ . Postponing fuller accounts to further research, we mention at least the following important connection. In [20], Vermeer constructs the basically disconnected cover of  $X$  (in the more general case of a completely regular Hausdorff space) through an inverse limit [20, Theorem 4.4]. The inverse limit is defined by means of an auxiliary space  $\Lambda_1(X)$  — itself a cover of  $X$  — for which see in particular [20, Theorem 3.5]. Now, the results by Hager et al. in [14, Section 3] entail amongst other things that, in our notation, there is a homeomorphism  $\text{Max } \mathcal{P}(C(X)) \cong \Lambda_1(X)$ . By Lemma 4.1 we conclude  $\Lambda_1(X) \cong w_{C(X)} Z_{C(X)}$ . We thus obtain a description of Vermeer’s cover  $\Lambda_1(X)$  as the Wallman compactification of the *minimal* spectral space of  $C(X)$  described in (9–11) above. Contrast Vermeer’s [20, Theorem 3.5], which describes  $\Lambda_1(X)$  using regularised zero sets in the *maximal* spectral space  $X$  of  $C(X)$ . Further, the space  $Z_{C(X)}$  is compact if, and only if,  $X$  is *cozero-complemented*, i.e. the lattice of cozero sets of  $X$  is complemented; this is well-known and also follows from Lemma 3.2. In this case we have  $w_{C(X)} Z_{C(X)} \cong Z_{C(X)}$ . Consequently,  $\Lambda_1(X) \cong Z_{C(X)}$  — when  $X$  is cozero-complemented, Vermeer’s  $\Lambda_1(X)$  is the minimal spectral space of  $C(X)$ , and further work reveals Vermeer’s cover  $\Lambda_1(X) \rightarrow X$  in [20] as the “push-up” map  $\lambda: Z_{C(X)} \rightarrow X$  in (1–2).

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